

Holographic entanglement entropy in the nonconformal medium

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ABSTRACT

We investigate holographically the entanglement entropy of a nonconformal medium whose dual geometry is described by an Einstein-Maxwell-dilaton theory. Due to an additional conserved charge corresponding to the number operator, its thermodynamics can be represented in a grandcanonical or canonical ensemble. We study thermodynamics in both ensembles by using the holographic renormalization and the entanglement entropy of a nonconformal medium. After defining the entanglement chemical potential which unlike the entanglement temperature has a nontrivial size dependence, we find that the entanglement entropy of a small subsystem satisfies the relation resembling the first law of thermodynamics in a medium. Furthermore, we study the entanglement entropy change in the nonconformal medium caused by the excitation of the ground state and by the global quench corresponding to the insertion of particles.

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1 Introduction

For the last decade, there have been a huge amount of efforts to understand strongly interacting systems via the AdS/CFT correspondence [1]. This new concept allowed us to study various microscopic as well as macroscopic properties of a conformal field theory (CFT) in the strong coupling regime, for example, 4-dimensional $\mathcal{N} = 4$ super Yang-Mills theory [2, 3, 4] or 3-dimensional $\mathcal{N} = 6$ Chern-Simons gauge theory [5, 6]. Its dual geometry is usually represented as an asymptotic anti de-Sitter (AdS) space with a proper compact manifold. These researches have been further generalized to the relativistic nonconformal and nonrelativistic field theory [8]-[26]. Accumulating knowledges on those nonconformal examples would be important in understanding the underlying structure of the gauge/gravity duality in depth and in applying them to more realistic physical phenomena of quantum chromodynamics and condensed matter system. Moreover, in order to figure out the quantum aspect of such systems, the entanglement entropy becomes an important issue [27]-[45]. In this paper, we will study the entanglement entropy and its thermodynamics-like behavior in a medium holographically.

For regarding a nonconformal theory, we should violate a scaling symmetry of the dual geometry which can be realized by adding a scalar field called dilaton. Then, the resulting geometry does not allow an asymptotic AdS space as a solution. Let us take into account an Einstein-Maxwell-dilaton theory. This gravity theory permits two different types of the generalized black brane solution. One is a charged dilatonic black brane whose dual theory is mapped to a relativistic nonconformal theory with matter, while the other is an uncharged black brane dual to a generalized Lifshitz geometry where the Lifshitz scaling symmetry is broken [13]. The first corresponds to the deformed Reissner-Nordström AdS black brane with a nontrivial dilaton profile. The gauge field plays a different role in those two examples. For a charged dilatonic black brane, the bulk gauge field provides an additional conserved charge representing one of the black brane hairs, so that its dual is clearly interpreted as the number density operator of the matter. In the generalized Lifshitz theory, the bulk field is not free and does not provide a new black brane hair. Instead, it breaks the boost symmetry and generates the anisotropy between time and spatial coordinates. This is why the Lifshitz-type uncharged black brane appears [8, 9, 11, 13]. Anyway, since we are interested in the nonconformal medium, we focus on a charged dilatonic black brane from now on.

In general, it is not easy to calculate the entanglement entropy of an interacting quantum field theory (QFT). However, the gauge/gravity duality can shed light on studying the entanglement entropy even in a strong coupling regime. In [27, 28, 29], it was shown that the holographic entanglement entropy proportional to the area of the minimal surface exactly reproduces the known results in a 2-dimensional CFT [46]. This work was further generalized to the higher dimensional cases. In an IR limit, the holographic entanglement entropy in a black hole geometry reduces to the well-known Bekenstein-Hawking entropy. Meanwhile, in a small subsystem corresponding to a UV limit it describes the entanglement entropy of excited states [47]. In spite of the fact that the entanglement temperature

is different from the real temperature of the system, the holographic entanglement entropy satisfies the thermodynamics-like relation. In a dual CFT, the entanglement temperature is proportional to the inverse of a subsystem size, $T_E \sim 1/l$. This is also true for a relativistic nonconformal QFT dual to a hyperscaling violation geometry [47]-[55]. This fact implies that the size dependence of the entanglement temperature is independent of details of the theory and the entangling surface. From now on, we say that the entanglement temperature is universal because it always has the same form in a relativistic dual QFT¹.

Is this still true in a medium? To answer this question, let us first think of thermodynamics. In a medium, there exists an additional conserved quantity corresponding to the number of particles. So the first law of thermodynamics is modified into the form including the particle number. For the entanglement entropy to satisfy such a modified thermodynamic relation, one should define a new variable representing the chemical potential which we will call the entanglement chemical potential. Like the entanglement temperature, it is different from the chemical potential defined in thermodynamics. Due to the modification of the thermodynamic relation and the new conserved charge in the medium, we cannot easily answer the previous question and furthermore new issues appear. Does the entanglement entropy in a medium follow the modified thermodynamics-like relation? If so, does the entanglement temperature still show the same universality? Lastly, does the newly defined entanglement chemical potential have a universal form independent of the details of the theory? One of goals in this paper is to clarify them. We find that the entanglement entropy in a medium follows the modified first thermodynamics-like relation and that the entanglement temperature still remains universal. However, we show that the size dependence of the entanglement chemical potential nontrivially relies on the nonconformality. Finally, we consider the uniform insertion of particles at zero temperature which can be regarded as a global quench deforming the original theory. Since this global quench modifies the quantum states of the system, the entanglement entropy changes. Under such a global quench, we calculate the change of the entanglement entropy quantitatively.

The rest of paper is organized as follows. In Sec. 2, we study a charged black brane solution of an Einstein-Maxwell-dilaton gravity which is dual to a nonconformal medium. From this solution, we calculate the thermodynamic properties in a grandcanonical and canonical ensemble. In Sec. 3, its entanglement entropy in a small subsystem is taken into account. Due to an additional conserved quantity, the entanglement chemical potential is newly defined. Using it we show that the entanglement entropy follows the first thermodynamics-like relation and that the entanglement chemical potential has a nontrivial size dependence relying on the nonconformality. In addition, we also investigate the change of the entanglement entropy under the global quench corresponding to the uniform insertion of particles. We finish our work with some concluding remarks in Sec. 4.

¹In more general nonrelativistic cases with a dynamical exponent, the size dependence of the entanglement temperature is further generalized to $T_E \sim 1/l^z$ [47].

2 Charged dilatonic black brane in the Einstein-Maxwell-dilaton theory

Let us consider the following Einstein-Maxwell-dilaton gravity [13, 14, 16]

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R - 2(\partial\phi)^2 - \frac{e^{2\alpha\phi}}{4} F_{\mu\nu} F^{\mu\nu} - 2\Lambda e^{\eta\phi} \right], \quad (1)$$

where Λ denotes a negative cosmological constant. This theory provides several different geometric solutions. If ϕ is a constant and $F_{\mu\nu} = 0$, the simplest solution is given by a 4-dimensional AdS space. It can be generalized to a Schwarzschild AdS (SAdS) black brane in the Poincare patch (or SAdS black hole in the global patch). The SAdS black brane is characterized by one parameter called the black brane mass. Turning on the gauge flux, SAdS black brane is further generalized to a Reissner-Nordström AdS (RNAdS) black brane with two hairs, mass and charge. The asymptote of all these solutions is described by an AdS geometry.

When ϕ has a nontrivial profile, the previous geometries are not solutions anymore. In this case, the solutions of the Einstein-Maxwell-dilaton theory are classified as follows. If $F_{\mu\nu} = 0$, the Einstein-Maxwell-dilaton theory reduces to an Einstein-dilaton theory, which allows a hyperscaling violation geometry [14]-[26]. Since the overall factor of the hyperscaling violation metric breaks the scaling symmetry, its asymptote is not an AdS space. In spite of breaking of the scaling symmetry, the rotation and translation symmetries of the boundary space still survive. This fact implies that its dual field theory corresponds to a relativistic nonconformal QFT. In general, the hyperscaling violation geometry has a naked singularity at the center which may indicate the instability or incompleteness of the theory. This fact indicates that the dual QFT is IR incomplete. To avoid this problem, one can regard the black brane geometry. Analogous to a SAdS black brane, the hyperscaling violation geometry can be easily generalized to an uncharged black brane where the singularity is hidden behind the horizon. In the dual field theory at finite temperature, there is no IR incompleteness because the Hawking temperature plays an effective IR cutoff. Even in this case, the zero temperature limit still remains problematic. Another way to get rid of the IR incompleteness is to take into account a medium which is dual of a charged black brane geometry. In this case, the dual QFT has an IR fixed point at which the dual theory effectively becomes a $1+1$ -dimensional CFT. As a result, a QFT with matter dual to a charged dilatonic black brane is free from the IR incompleteness even at zero temperature. In order to obtain a charged dilatonic black brane solution, let's turn on the gauge flux with an appropriate parameter α . Then, we can expect that an uncharged black brane solution of the Einstein-dilaton gravity [13] is modified into a charged one with two black brane hairs. It is true only for a specific value of α . For general α , intriguingly, there exists another uncharged black brane solution in which the boost symmetry as well as the scaling symmetry are broken. Thus the time and spatial coordinate behave differently [12, 13]. This is the generalization of the well-known Lifshitz geometry [8]. In this paper, we concentrate on a QFT dual to a charged dilatonic black brane and

investigate its quantum aspects described by a holographic entanglement entropy.

The equations of motion for the Einstein-Maxwell-dilaton theory are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + g_{\mu\nu}\Lambda e^{\eta\phi} = 2\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}(\partial\phi)^2 + \frac{e^{2\alpha\phi}}{2}F_{\mu\lambda}F_\nu{}^\lambda - \frac{e^{2\alpha\phi}}{8}g_{\mu\nu}F^2, \quad (2)$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) = \frac{\eta\Lambda}{2}e^{\eta\phi} + \frac{\alpha}{8}e^{2\alpha\phi}F^2, \quad (3)$$

$$0 = \partial_\mu(\sqrt{-g}e^{2\alpha\phi}F^{\mu\nu}). \quad (4)$$

In order to solve these equations, we take a logarithmic dilaton profile

$$\phi(r) = \phi_0 - \phi_1 \log r, \quad (5)$$

where ϕ_0 and ϕ_1 are two integration constants. Since ϕ_0 can be absorbed into the cosmological constant, we can set $\phi_0 = 0$ without loss of generality. Now, we consider the following metric ansatz for a charged dilatonic black brane

$$ds^2 = -g(r)^2 f(r) dt^2 + \frac{dr^2}{g(r)^2 f(r)} + h(r)^2 (dx^2 + dy^2), \quad (6)$$

with

$$g(r) = g_0 r^{g_1}, \quad h(r) = h_0 r^{h_1}. \quad (7)$$

Here the diffeomorphism allows us to set $g_0 = h_0 = 1$. When we turn on a time-component gauge field A_t only, the electric field satisfying (4) is given by

$$F_{rt} = \frac{q}{h(r)^2} e^{-2\alpha\phi}. \quad (8)$$

Note that a charged black brane we consider is the generalization of an uncharged black brane studied in the Einstein-dilaton theory [25], which preserves the boundary Lorentz symmetry. Since the bulk gauge field, following the gauge/gravity duality, is related to the matter, the dual field theory of a charged black brane corresponds to a relativistic nonconformal QFT with the matter. In order to preserve the boundary Lorentz symmetry, $g_1 = h_1$ should be satisfied. Furthermore, only when $\alpha = -\eta/2$, there exists a charged black brane solution satisfying all equations of motion. In this case, the integration constants are determined as

$$g_1 = \frac{4}{4 + \eta^2}, \quad \phi_1 = \frac{2\eta}{4 + \eta^2} \quad \text{and} \quad \Lambda = -\frac{4(12 - \eta^2)}{(4 + \eta^2)^2}, \quad (9)$$

and the black brane factor is given by

$$f(r) = 1 - \frac{m}{r^a} + \frac{b}{r^c}, \quad (10)$$

with

$$a = \frac{12 - \eta^2}{4 + \eta^2}, \quad b = \frac{4 + \eta^2}{16} Q^2 \quad \text{and} \quad c = a + 1 = \frac{16}{4 + \eta^2}, \quad (11)$$

where m and Q denote two black brane hairs. In the $\eta = 0$ limit, this charged dilatonic black brane reduces to an RNAdS black brane and the scaling symmetry is restored [25]. The near horizon geometry of the charged dilatonic black brane reduces to $AdS_2 \times R^2$, which is independent of the nonconformality, η , and shows the existence of an IR fixed point effectively described by a 1 + 1-dimensional CFT. Above we used the nonconformality parameter to clarify the nonconformal effect, which is also related to the hyperscaling violation exponent [26]

$$\theta = -\frac{2\eta^2}{4 - \eta^2}. \quad (12)$$

As shown in [56], the holographic renormalization together with regularity conditions of bulk fields provide a boundary stress tensor consistent with the black brane thermodynamics. The regularity of the metric requires that there is no conical singularity at the horizon and yields the Hawking temperature

$$T_H = \frac{12 - \eta^2}{4(4 + \eta^2)\pi} \left(1 - \frac{(4 + \eta^2)^2}{16(12 - \eta^2)} \frac{Q^2}{r_h^{\frac{16}{4+\eta^2}}} \right) r_h^{\frac{4-\eta^2}{4+\eta^2}}. \quad (13)$$

From the Maxwell equation, the time component of the vector field A_t is determined as

$$A_t = 2\kappa^2\mu - \frac{Q}{r}, \quad (14)$$

where μ is an integration constant interpreted as a chemical potential. The regularity of the vector field norm at the horizon gives rise to the relation between the chemical potential and the particle number

$$N = 2\kappa^2 V_2 \mu r_h, \quad (15)$$

where $N = QV_2$.

2.1 Holographic renormalization of the grandcanonical ensemble

Let us consider the holographic renormalization of the Einstein-Maxwell-dilaton theory, which provides direct interpretation of the boundary energy-momentum tensor as thermodynamic quantities. With an Euclidean signature, the Einstein-Maxwell-dilaton action is rewritten as

$$S_E = \int d^4x \mathcal{L}_D, \quad (16)$$

with

$$\mathcal{L}_D = -\frac{1}{2\kappa^2} \sqrt{g} \left[R - 2(\partial\phi)^2 - \frac{e^{2\alpha\phi}}{4} F_{\mu\nu} F^{\mu\nu} - 2\Lambda e^{\eta\phi} \right], \quad (17)$$

where the Euclidean metric is given by

$$ds_E^2 = r^{2g_1} f(r) dt^2 + \frac{dr^2}{r^{2g_1} f(r)} + r^{2g_1} (dx^2 + dy^2), \quad (18)$$

and the Euclidean vector field becomes

$$A_\tau = -i \left(2\kappa^2 \mu - \frac{Q}{r} \right). \quad (19)$$

These metric and vector field together with the dilaton field in (5) satisfy the Euclidean equations of motion.

In order to evaluate the on-shell gravity action, we should add several boundary terms. The first is the Gibbons-Hawking term which is required to define the metric variation well

$$S_{GH} = \frac{1}{\kappa^2} \int_{\partial\mathcal{M}} d^3x \sqrt{\gamma} \Theta, \quad (20)$$

where γ_{ij} indicates an induced metric on the boundary and the extrinsic curvature is given, in terms of the unit normal vector n_μ , by

$$\Theta_{\mu\nu} = -\frac{1}{2} (\nabla_\mu n_\nu + \nabla_\nu n_\mu). \quad (21)$$

The second boundary term is a local counter term which is needed to make the on-shell action finite at the boundary. The correct counter term is

$$S_{ct} = \frac{8}{(4 + \eta^2)\kappa^2} \int_{\partial\mathcal{M}} d^3x \sqrt{\gamma} e^{\eta\phi/2}. \quad (22)$$

This term is the same as the one used in the holographic renormalization of the Einstein-dilaton theory [26]. Since the vector field of the charged black brane does not generate new divergence at the UV regime, no additional counter is required [56]. If we impose a Dirichlet boundary condition on the vector field at the asymptotic boundary, it fixes the chemical potential. In this case, all physical quantities should be represented as functions of the chemical potential and the on-shell gravity action, in the dual QFT point of view, is proportional to the grand potential of a grandcanonical ensemble. On the other hand, imposing a Neumann boundary condition instead of a Dirichlet boundary condition is related to choose a canonical ensemble and requires an additional boundary term corresponding to the Legendre transformation.

The grand potential with a Dirichlet boundary condition leads to

$$\begin{aligned} \Omega(T_H, \mu, V_2) &= T_H (S_E + S_{GH} + S_{ct}) \\ &= -\frac{(4 - \eta^2) V_2}{2(4 + \eta^2)\kappa^2} r_h^{(12 - \eta^2)/(4 + \eta^2)} \left(1 + \frac{(4 + \eta^2)\kappa^4}{4} \frac{\mu^2}{r_h^{2(4 - \eta^2)/(4 + \eta^2)}} \right), \end{aligned} \quad (23)$$

where V_2 denotes the spatial volume at the boundary and the horizon r_h becomes an implicit function of T_H , μ and V_2 from (13) and (15). The boundary energy-momentum tensor, which is obtained by varying the on-shell gravity action with respect to the boundary metric

$$T^i_j \equiv \lim_{r_0 \rightarrow \infty} \left(-2 \gamma^{ik} \int d^3x \frac{\delta \mathcal{L}_D}{\delta \gamma^{kj}} \right), \quad (24)$$

reads

$$\begin{aligned}
E &= T_0^0 \\
&= \frac{4V_2}{(4+\eta^2)\kappa^2} r_h^{(12-\eta^2)/(4+\eta^2)} \left(1 + \frac{(4+\eta^2)\kappa^4}{4} \frac{\mu^2}{r_h^{2(4-\eta^2)/(4+\eta^2)}} \right), \\
P &= -\frac{T_1^1}{V_2} = -\frac{T_2^2}{V_2} \\
&= \frac{(4-\eta^2)}{2(4+\eta^2)\kappa^2} r_h^{(12-\eta^2)/(4+\eta^2)} \left(1 + \frac{(4+\eta^2)\kappa^4}{4} \frac{\mu^2}{r_h^{2(4-\eta^2)/(4+\eta^2)}} \right). \tag{25}
\end{aligned}$$

From this result, one can see that the grand potential is related to pressure

$$\Omega = -PV_2. \tag{26}$$

From the exact differential relation, the canonical conjugate variables of the fundamental variables, T_H , μ and V_2 , are evaluated to

$$S = -\left. \frac{\partial \Omega}{\partial T_H} \right|_{\mu, V_2} = \frac{2\pi V_2}{\kappa^2} r_h^{8/(4+\eta^2)}, \tag{27}$$

$$N = -\left. \frac{\partial \Omega}{\partial \mu} \right|_{T_H, V_2} = 2\kappa^2 V_2 r_h \mu, \tag{28}$$

$$P = -\left. \frac{\partial \Omega}{\partial V_2} \right|_{T_H, \mu} = \frac{(4-\eta^2)}{2(4+\eta^2)\kappa^2} r_h^{(12-\eta^2)/(4+\eta^2)} \left(1 + \frac{(4+\eta^2)\kappa^4}{4} \frac{\mu^2}{r_h^{2(4-\eta^2)/(4+\eta^2)}} \right). \tag{29}$$

Since the conjugate variable of temperature is the entropy, S denotes the renormalized thermal entropy derived from the renormalized grand potential. Intriguingly, this renormalized thermal entropy coincides with the Bekenstein-Hawking entropy. The particle number in (28) is in agreement with the regularity of the vector field. As a consequence, the holographic renormalization results are perfectly matched to those of the charged black brane thermodynamics in (A.11). The equation of state parameter of this system is given by

$$\omega = \frac{PV_2}{E} = \frac{1}{2} - \frac{\eta^2}{8}. \tag{30}$$

Since the equation of parameter is independent of the chemical potential, it is the same as that obtained in the Einstein-dilaton theory [26]. Here η indicates the nonconformality representing the deviation from the CFT.

According to the AdS/CFT correspondence, the conformal dimension of the dual operator for a bulk p -form field in AdS_{d+1} is determined by [2, 3]

$$(\Delta + p)(\Delta + p - d) = m^2, \tag{31}$$

where the largest value of Δ corresponds to the conformal dimension of the dual operator. This relation says that for $\eta = 0$ the dual operator of a massless bulk vector field ($p = 1$ and $d = 3$) has a

conformal dimension 2. In general, we may consider many different conformal dimension 2 operators composed of scalars or fermions. One example we are interested in is the operator composed of two fermions, $\mathcal{O}_\mu = \bar{\psi}\gamma_\mu\psi$. Since a fermion in a 2 + 1-dimensional conformal field theory has a conformal dimension 1, \mathcal{O}_μ can be a dual operator. In this case, similar to the 5-dimensional RNAdS black brane [56], we can regard the boundary value of a time-component gauge field $A_t(z=0)$ as the chemical potential. For a general η , the relation between the bulk field and boundary operator is not clear. However, if it is regarded as the nonconformal deformation from the conformal field theory, it may be possible to generalize the AdS/CFT correspondence to the nonconformal case. Here, we just assume that there exists such a generalization.

The action form we consider openly appears in the string theory with specific value of η [58, 59, 60, 61]. In this case, the bulk scalar field appears as a dilaton in the string theory and the boundary value of the dilaton field is identified with the gauge coupling of the dual QFT. In the holographic point of view, the nontrivial dilaton profile implies the nontrivial gauge coupling depending on the energy scale. For $\eta = 0$, since the dual theory is conformal, the dilaton field becomes trivial. On the other hand, (30) shows an explicit nonconformality for a general η . It would be interpreted as the effect of an irrelevant deformation or interaction because it affects on the UV behavior. More precisely, the deviation from the conformal theory can be read from the trace of the energy-momentum tensor. Taking the trace of the stress tensor in (25), yields

$$T^i_i = E - 2P = \frac{\eta^2}{(4 + \eta^2)\kappa^2} r_h^{(12-\eta^2)/(4+\eta^2)} + \frac{\eta^2\kappa^2}{4} \mu^2 r_h. \quad (32)$$

The first term shows the effect of a nonconformal interaction, whereas the second is the effect of the matter. These effects disappear in the conformal limit, $\eta \rightarrow 0$, as expected. Intriguingly, for $\eta^2 = 4$ the energy and pressure are reduced to

$$E = \frac{V_2}{2\kappa^2} (1 + 2\kappa^4\mu^2) r_h \quad \text{and} \quad P = 0, \quad (33)$$

which implies that the dual system is composed of pressureless particles, the so-called dust.

Now let us consider the zero temperature behavior. In the extremal case ($T_H = 0$) except the dust, the horizon in terms of the chemical potential is given by

$$r_h = \left(\frac{(4 + \eta^2)\kappa^2\mu}{2\sqrt{12 - \eta^2}} \right)^{(4+\eta^2)/(4-\eta^2)}. \quad (34)$$

If the dual matter is fermionic, we may regard a Fermi surface. If there exists such a Fermi surface even in the strong coupling regime, the Fermi surface energy at zero temperature can be identified with the chemical potential, $\epsilon_F = \mu$. From (28) the Fermi surface energy is proportional to the fermion number density, $n = N/V_2$,

$$\epsilon_F \sim n^{(4-\eta^2)/8}. \quad (35)$$

For the conformal case, the Fermi surface energy is proportional to \sqrt{n} which is similar to that of 2 + 1-dimensional free fermions, although we cannot directly compare them.

In the dust case with $\eta^2 = 4$, (34) becomes singular so that we cannot apply it directly. Instead, we should look at the metric which becomes simple for the dust

$$g_{tt} = - \left(r - m + \frac{Q^2}{2r} \right). \quad (36)$$

For $m \geq \sqrt{2}Q$, it has two horizons

$$r_{\pm} = \frac{m \pm \sqrt{m^2 - 2Q^2}}{2}, \quad (37)$$

and the case saturating $m = \sqrt{2}Q$ gives rise to the extremal limit corresponding to the zero temperature. The Hawking temperature reads in terms of the chemical potential

$$T_H = \frac{1}{4\pi} (1 - 2\kappa^4 \mu^2). \quad (38)$$

At zero temperature, the chemical potential is given by

$$\mu = \frac{1}{\sqrt{2}\kappa^2}. \quad (39)$$

Using this relation, the horizon is determined only by m

$$r_h = \frac{m}{2}, \quad (40)$$

where m still remains as a free parameter. The number density of the dust at zero temperature yields

$$n = \sqrt{2}m, \quad (41)$$

where (39) was used. This result shows that there is no direct relation between Fermi surface energy and momentum.

2.2 Holographic renormalization of the canonical ensemble

As mentioned previously, if one imposes a Neumann boundary condition on the vector field, the charge density Q is fixed. In this case, the dual system is described by a canonical ensemble. To see this, let us vary the action with respect to the vector field. Then, it generally generates a nontrivial boundary term, the so-called Neumannizing term [10],

$$\delta S_E = \frac{1}{2\kappa^2} \int_{\partial\mathcal{M}} d^3x \sqrt{g} e^{2\alpha\phi} g^{rr} g^{\tau\tau} F_{r\tau} \delta A_\tau. \quad (42)$$

When imposing the Dirichlet boundary condition, this term automatically vanishes because of $\delta A_\tau = 0$. However, the Neumann boundary condition cannot get rid of the Neumannizing term. In order to remove it, we should add a new boundary term

$$S_{bd} = \frac{1}{2\kappa^2} \int_{\partial\mathcal{M}} d^3x A_\mu J^\mu, \quad (43)$$

with $A_\mu = \{A_\tau, 0, 0\}$ and $J^\mu = \{iQ, 0, 0\}$. The existence of this boundary term indicates that the boundary condition changes from Dirichlet to Neumann boundary condition

$$e^{2\alpha\phi} g^{rr} g^{\tau\tau} F_{r\tau} \Big|_{r=\infty} = -iQ. \quad (44)$$

In the dual field theory, it is nothing but the Legendre transformation between the grandcanonical and canonical ensembles. This fact has been crucially used in studying the phase diagram of the holographic QCD [62].

When the Neumann boundary condition is imposed, the renormalized action of a canonical ensemble is described by

$$S_{can} = S_E + S_{GH} + S_{ct} + S_{bd}, \quad (45)$$

and all quantities should be functions of T_H , N and V_2 . The on-shell gravity action leads to the following the free energy

$$\begin{aligned} F &= \frac{S_{can}}{\beta} \\ &= -\frac{(4-\eta^2)V_2}{2(4+\eta^2)\kappa^2} r_h^{(12-\eta^2)/(4+\eta^2)} \left(1 - \frac{(12+\eta^2)(4+\eta^2)}{16(4-\eta^2)V_2^2} \frac{N^2}{r_h^{16/(4+\eta^2)}} \right), \end{aligned} \quad (46)$$

where r_h is given by an implicit function of T_H , N and V_2 . The internal energy and pressure are from the boundary energy-momentum tensor

$$\begin{aligned} E &= \frac{4V_2}{(4+\eta^2)\kappa^2} r_h^{(12-\eta^2)/(4+\eta^2)} \left(1 + \frac{4+\eta^2}{16V_2^2} \frac{N^2}{r_h^{16/(4+\eta^2)}} \right), \\ P &= \frac{4-\eta^2}{2(4+\eta^2)\kappa^2} r_h^{(12-\eta^2)/(4+\eta^2)} \left(1 + \frac{4+\eta^2}{16V_2^2} \frac{N^2}{r_h^{16/(4+\eta^2)}} \right). \end{aligned} \quad (47)$$

Like the grandcanonical ensemble case, all these results are in agreement with the thermodynamic results of the charged black brane in (A.7).

3 Holographic entanglement entropy in a medium

The entanglement entropy has been paid much attention to study quantum aspects of the QCD and condensed matter system. Recently, it was conjectured following the AdS/CFT correspondence that the entanglement entropy of a strongly interacting system can be understood by investigating a holographic minimal surface in the dual AdS geometry [27, 28, 29]. This idea on the holographic entanglement entropy is further generalized to non-AdS geometries dual to nonconformal field theories [29, 26]. In their subsequent works, intriguingly, it was shown holographically that the entanglement entropy in a small subsystem characterized by l reveals the first thermodynamics-like relation. That is, the entanglement entropy of excited states follows the first law of thermodynamics

$$T_E \Delta S_E = \Delta E, \quad (48)$$

where ΔE indicates the increased energy and T_E is called the entanglement temperature. The entanglement temperature is different from the thermal temperature and is given by [47]

$$T_E \sim \frac{c}{l}. \quad (49)$$

Although a constant, c , depends on details of the theory and the shape of the entangling surface, the form of the entanglement temperature in (49) is independent of them. From this point of view, we say that the entanglement temperature is universal.

This universality has been also checked in a relativistic nonconformal field theories without matter [26, 47]. Unlike an uncharged black brane cases, a charged black brane has an additional conserved quantity which usually plays an important role in thermodynamics. Depending on ensemble we choose, it becomes a particle number or chemical potential. When the volume is fixed, the additional quantity modifies the first law of thermodynamics into

$$dE = TdS + \mu dN. \quad (50)$$

Similar to a thermal system, one can expect that the first thermodynamics-like law of the entanglement entropy should also be modified in a medium. In this section, we will investigate the entanglement entropy in a medium and check whether it satisfies the thermodynamics-like relation.

3.1 Holographic entanglement entropy in a strip

Let us consider the holographic entanglement entropy of a thin strip [26, 27, 28, 57]. The dual field theory of the previous charged black brane is given by a $2 + 1$ -dimensional relativistic nonconformal theory, so we take a subsystem as a 2-dimensional thin strip and evaluate the entanglement entropy contained in it. First, we assume that the dual field theory lives in a L^2 spatial volume

$$-\frac{L}{2} \leq x \leq \frac{L}{2} \quad \text{and} \quad -\frac{L}{2} \leq y \leq \frac{L}{2}, \quad (51)$$

which is a total system we consider. Now, let us divide this system into two subsystems, A and A^c , and take the area of the subsystem A as a thin strip

$$-\frac{l}{2} \leq x \leq \frac{l}{2} \quad \text{and} \quad -\frac{L}{2} \leq y \leq \frac{L}{2}, \quad (52)$$

where the width of the strip, l , is smaller than L . Following the holographic entanglement prescription, the entanglement entropy can be evaluated by calculating the area of the minimal surface in the dual geometry, whose boundary should coincide with the entangling surface of the strip. The induced metric, h_{ij} , on the minimal surface becomes from (6)

$$ds^2 = \left(\frac{r'^2}{r^{2g_1} f(r)} + r^{2g_1} \right) dx^2 + r^{2g_1} dy^2, \quad (53)$$

where the prime indicates a derivative with respect to x . Then, the action governing the area of the minimal surface reduces to

$$A = L \int_{-l/2}^{l/2} dx \sqrt{\frac{r'^2}{f(r)} + r^{4g_1}}. \quad (54)$$

Due to the parity invariance under $x \rightarrow -x$, the minimal surface should have a turning point at $x = 0$ which gives rise to the minimum of r . If we denote it by r_* , the range spanned by the minimal surface is restricted to $r_* \leq r < r_{UV}$, where r_{UV} denotes a UV cutoff.

For convenience, let us introduce dimensionless variables scaled by r_*

$$z = \frac{r}{r_*}, \quad z_h = \frac{r_h}{r_*}, \quad \tilde{m} = \frac{m}{r_*^3}, \quad \tilde{b} = \frac{b}{r_*^4} \text{ and } z_{UV} = \frac{r_{UV}}{r_*}. \quad (55)$$

Then, the black brane factor can be rewritten as

$$f(z) = 1 - \left(1 + \frac{\tilde{b}}{z_h^{a+1}}\right) \frac{z_h^a}{z^a} + \frac{\tilde{b}}{z^{a+1}}, \quad (56)$$

where \tilde{m} is given by a function of \tilde{b} and z_h . For a small l , we should take into account the case, $z_h \ll 1$. In terms of dimensionless variables, the Hawking temperature becomes

$$T_H = \frac{1}{4\pi z_h r_*} \left(a - \frac{\tilde{b}}{z_h^{a+1}}\right). \quad (57)$$

In order to define temperature well, \tilde{b}/z_h^{a+1} should be smaller than $a \sim \mathcal{O}(1)$. The saturation of this relation corresponds to the extremal limit, in other words, the zero temperature limit.

The system we consider is invariant under the translation in the x -direction. If regarding x as a time coordinate, the Hamiltonian is conserved. From it, the width of the strip can be represented as an integral form

$$l = \frac{2}{r_*^{2g_1-1}} \int_1^{z_{UV}} dz \frac{1}{z^{2g_1} \sqrt{z^{4g_1}-1}} \frac{1}{\sqrt{f(z)}}. \quad (58)$$

Here the small l corresponds to the large r_* because $2g_1 - 1 > 0$. Expanding $1/\sqrt{f(z)}$ and integrating (58) order by order, r_* is determined as a function of l

$$r_* = c_0 l^{-\frac{1}{2g_1-1}} + c_1 l^{\frac{a-1}{2g_1-1}} + c_2 l^{\frac{a}{2g_1-1}} + \mathcal{O}\left(l^{\frac{a+1}{2g_1-1}}\right), \quad (59)$$

where

$$c_0 = \frac{\pi^{\frac{1}{4g_1-2}} \Gamma\left(1 - \frac{1}{4g_1}\right)^{\frac{1}{2g_1-1}}}{2^{\frac{1}{2g_1-1}} g_1^{\frac{1}{2g_1-1}} \Gamma\left(\frac{3}{2} - \frac{1}{4g_1}\right)^{\frac{1}{2g_1-1}}},$$

$$\begin{aligned}
c_1 &= \frac{2^{\frac{a-2g_1}{2}} g_1^{\frac{a-1}{2g_1-1}} (r_h^{a+1} + b) \Gamma\left(\frac{3}{2} - \frac{1}{4g_1}\right)^{\frac{2g_1+a-2}{2g_1-1}} \Gamma\left(\frac{a-1}{4g_1} + 1\right)}{(2g_1 - 1) \pi^{\frac{a-1}{4g_1-2}} r_h \Gamma\left(1 - \frac{1}{4g_1}\right)^{\frac{2g_1+a-2}{2g_1-1}} \Gamma\left(\frac{6g_1+a-1}{4g_1}\right)}, \\
c_2 &= -\frac{2^{\frac{a-2g_1+1}{2}} b g_1^{\frac{a}{2g_1-1}} \Gamma\left(\frac{3}{2} - \frac{1}{4g_1}\right)^{\frac{2g_1+a-1}{2g_1-1}} \Gamma\left(\frac{a}{4g_1} + 1\right)}{(2g_1 - 1) \pi^{\frac{a}{4g_1-2}} \Gamma\left(1 - \frac{1}{4g_1}\right)^{\frac{2g_1+a-1}{2g_1-1}} \Gamma\left(\frac{a}{4g_1} + \frac{3}{2}\right)}.
\end{aligned} \tag{60}$$

Substituting these results into the action in (54), the holographic entanglement entropy reads perturbatively

$$S_E \equiv \frac{2\pi A}{\kappa^2} = \frac{4\pi L}{\kappa^2} r_{UV} + s_0 l^{-\frac{1}{2g_1-1}} + s_1 l^{\frac{a-1}{2g_1-1}} + s_2 l^{\frac{a}{2g_1-1}} + \dots, \tag{61}$$

where ellipsis implies higher order terms and coefficients, s_0 , s_1 and s_2 , are given by

$$\begin{aligned}
s_0 &= -\sqrt{\pi} L \frac{2 c_0 \Gamma\left(1 - \frac{1}{4g_1}\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{4g_1}\right)}, \\
s_1 &= \sqrt{\pi} L \left\{ \frac{b \Gamma\left(\frac{a+4g_1-1}{4g_1}\right)}{(a-1) r_h c_0^{a-1} \Gamma\left(\frac{a+2g_1-1}{4g_1}\right)} + \frac{r_h^a \Gamma\left(\frac{a+4g_1-1}{4g_1}\right)}{(a-1) c_0^{a-1} \Gamma\left(\frac{a+2g_1-1}{4g_1}\right)} - \frac{2 c_1 \Gamma\left(1 - \frac{1}{4g_1}\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{4g_1}\right)} \right\}, \\
s_2 &= -\sqrt{\pi} L \left\{ \frac{b \Gamma\left(\frac{a}{4g_1} + 1\right)}{a c_0^a \Gamma\left(\frac{a}{4g_1} + \frac{1}{2}\right)} + \frac{2 c_2 \Gamma\left(1 - \frac{1}{4g_1}\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{4g_1}\right)} \right\}.
\end{aligned} \tag{62}$$

The first term in (61) shows the expected divergence of a $2+1$ -dimensional field theory, which is originated from the short range correlation near the boundary of the strip. The remainders are related to the long range correlation between the inside and outside of the strip.

3.2 Conformal medium with $\eta = 0$

Let us first consider the simplest conformal case. For $\eta = 0$, an Einstein-Maxwell-dilaton theory reduces to an Einstein-Maxwell theory and its geometric solution reduces to the well-known RNAdS black brane

$$ds^2 = -r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 (dx^2 + dy^2), \tag{63}$$

with

$$f(r) = 1 - \frac{m}{r^3} + \frac{Q^2}{4r^4}. \tag{64}$$

When $Q = 0$, it is further reduced to the SAdS black brane. If m also vanishes, the geometry becomes a pure AdS space which can be regarded as a zero temperature limit of the SAdS black brane. Its dual field theory is in the vacuum of a CFT.

If turning on m in a small l limit, the vacuum states become excited [47]. In this case, the holographic entanglement entropy of the subsystem A reads from (61)

$$S_E = \frac{4\pi r_{UV} L}{\kappa^2} - \frac{8\pi^2 \Gamma\left(\frac{3}{4}\right)^2 L}{\Gamma\left(\frac{1}{4}\right)^2 \kappa^2} \frac{1}{l} + \frac{\pi \Gamma\left(\frac{1}{4}\right)^2 L r_h^3}{16 \Gamma\left(\frac{3}{4}\right)^2 \kappa^2} l^2, \quad (65)$$

where $r_h = m^{1/3} = \frac{4\pi T_H}{3}$. The first two terms correspond to the vacuum entanglement entropy, while the last indicates the increased entanglement entropy for the excited state

$$\Delta S_E = \frac{\pi \Gamma\left(\frac{1}{4}\right)^2 L r_h^3}{16 \Gamma\left(\frac{3}{4}\right)^2 \kappa^2} l^2. \quad (66)$$

The energy used to excite the vacuum states of the subsystem A is from (47) with $Q = 0$

$$\Delta E = \frac{L}{\kappa^2} l r_h^3, \quad (67)$$

where the volume of the subsystem A is given by $V_2 = lL$. Defining the entanglement temperature T_E as

$$T_E = \frac{16 \Gamma\left(\frac{3}{4}\right)^2}{\pi \Gamma\left(\frac{1}{4}\right)^2} \frac{1}{l}, \quad (68)$$

above quantities satisfy the thermodynamics-like relation

$$\Delta E = T_E \Delta S_E. \quad (69)$$

As mentioned before, the entanglement temperature shows a universal behavior inversely proportional to the width of the strip.

Now, let us move on the medium case. In this case, since there exists an additional conserved charge, we cannot naively compare the entanglement entropy of a medium with that of the vacuum. In the conformal medium with $\eta = 0$ and $Q \neq 0$, the electric charge of the black brane, Q , is dual to the number density of particles in the dual CFT, either $Q = N/L^2$ for the total system or $Q = N_A/(lL)$ for the subsystem A . In terms of N_A , the lowest order terms of the entanglement entropy are rewritten as

$$S_E = \frac{4\pi L}{\kappa^2} r_{UV} - \frac{8\pi^2 \Gamma\left(\frac{3}{4}\right)^2 L}{\Gamma\left(\frac{1}{4}\right)^2 \kappa^2} \frac{1}{l} + \frac{\pi \Gamma\left(\frac{1}{4}\right)^2 [1 + N_A^2/(4l^2 L^2 r_h^4)] L r_h^3}{16 \Gamma\left(\frac{3}{4}\right)^2 \kappa^2} l^2 - \frac{\Gamma\left(\frac{1}{4}\right)^2 N_A^2}{40 \Gamma\left(\frac{3}{4}\right)^2 \kappa^2 L} l, \quad (70)$$

where r_h is regarded as an implicit function of T_H and N_A following the relation

$$T_H = \frac{3}{4\pi} r_h \left(1 - \frac{1}{12} \frac{N_A^2}{l^2 L^2 r_h^4} \right). \quad (71)$$

If one varies T_H with a fixed particle number, it yields the entanglement entropy variation

$$\begin{aligned} \Delta S_E|_{N_A} &= S_E(T_H, N_A) - S_E(0, N_A) \\ &= \frac{\pi \Gamma\left(\frac{1}{4}\right)^2 L}{16 \Gamma\left(\frac{3}{4}\right)^2 \kappa^2} \left[\left(1 + \frac{N_A^2}{4l^2 L^2 r_h^4} \right) r_h^3 - \left(1 + \frac{N_A^2}{4l^2 L^2 r_0^4} \right) r_0^3 \right] l^2, \end{aligned} \quad (72)$$

where $S_E(T_H, N_A)$ and $S_E(0, N_A)$ indicate the entanglement entropy of excited and ground states respectively and r_0 implies the horizon in the extremal limit, $r_0^4 = N_A^2/(12l^2L^2)$. Above only the third term in (70) contributes the entanglement entropy variation through r_h . In the same system, the increased energy comes from the holographic renormalization in (47)

$$\Delta E|_{N_A} = \frac{L}{\kappa^2} \left[\left(1 + \frac{N_A^2}{4l^2L^2r_h^4} \right) r_h^3 - \left(1 + \frac{N_A^2}{4l^2L^2r_0^4} \right) r_0^3 \right] l. \quad (73)$$

Intriguingly, this result shows that the increased energy is proportional to the variation of the entanglement entropy. When the entanglement temperature is defined as the previous one in (68), the conformal medium satisfies the following thermodynamics-like relation

$$\Delta E|_{N_A} = T_E \Delta S_E|_{N_A}. \quad (74)$$

Furthermore, the third term in (70) can be reinterpreted as $\frac{E}{T_E}$.

In the medium, the additional conserved charge provides another situation. Even when temperature of the system is not changed, the entanglement entropy can vary by adding particles. To see this, let us vary the particle number without changing temperature. From the relation in (71), no variation of temperature gives rise to a relation between variations of r_h and N_A

$$\Delta r_h = \frac{N_A}{6 \left[1 + N_A^2/(4l^2L^2r_h^4) \right] l^2L^2r_h^3} \Delta N_A. \quad (75)$$

From this, the change of the entanglement entropy reads

$$\Delta S_E|_{T_H} = \left(\frac{\pi \Gamma(\frac{1}{4})^2 \left[1 + N_A^2/(12l^2L^2r_h^4) \right] N_A}{16 \Gamma(\frac{3}{4})^2 \left[1 + N_A^2/(4l^2L^2r_h^4) \right] \kappa^2 L r_h} - \frac{\Gamma(\frac{1}{4})^2 N_A l}{20 \Gamma(\frac{3}{4})^2 \kappa^2 L} \right) \Delta N_A, \quad (76)$$

where $\Delta S_E|_{T_H}$ implies the entropy change at fixed T_H . and the variation of N_A can be understood as a global quench, as will be shown. The origin of the first term is the third term in (70), so it is related to the energy variation when varying the particle number. The last comes from the fourth term in (70), which is independent of the energy variation. Using the entanglement temperature, this relation can be rewritten as

$$T_E \Delta S_E|_{T_H} = \Delta E|_{T_H} - \mu_E|_{T_H} \Delta N_A, \quad (77)$$

where the increased energy, $\Delta E|_{T_H}$, and the entanglement chemical potential, $\mu_E|_{T_H}$, are given by

$$\begin{aligned} \Delta E|_{T_H} &= \frac{\left[1 + N_A^2/(12l^2L^2r_h^4) \right] N_A}{3 \left[1 + N_A^2/(4l^2L^2r_h^4) \right] \kappa^2 l L r_h} \Delta N_A, \\ \mu_E|_{T_H} &= \frac{4N_A}{5\pi\kappa^2 L}. \end{aligned} \quad (78)$$

In the conformal medium, the entanglement chemical potential is independent of the strip width, l , and proportional to the number of particles contained in the subsystem A . Due to the fact that the

entanglement temperature is universal, it is interesting to ask whether the entanglement chemical potential also shows a similar universal behavior in more general cases. In the next section, we will discuss on the universality of this entanglement chemical potential in the hyperscaling violation geometry. Combining (74) and (77), the total change of the entanglement entropy satisfies the first law of thermodynamics in the medium

$$T_E \Delta S_E = \Delta E - \mu_E \Delta N_A, \quad (79)$$

with

$$\begin{aligned} \Delta E &= \Delta E|_{N_A} + \Delta E|_{T_H}, \\ \Delta S_E &= \Delta S_E|_{N_A} + \Delta S_E|_{T_H}, \end{aligned} \quad (80)$$

where $\Delta E|_{N_A}$ and $\Delta S_E|_{N_A}$ come from the excitation of ground states, while the global quench gives rise to $\Delta E|_{T_H}$ and $\Delta S_E|_{T_H}$. Eq.(79) governs the most general case including both the excitation of ground states and the global quench.

In order to understand the above thermodynamics-like relation at zero temperature, let us first suppose that the matter of a medium resides in the ground state at zero temperature. Following the AdS/CFT correspondence, its number density can be reinterpreted as the dual operator of the background gauge field in the extremal RNAdS black brane geometry, $Q = \psi^+ \psi$. If the matter is distributed uniformly, Q can be regarded a global operator independent of the position. Then, the number of the matter in the medium is given by $N = \int d^2x Q = QL^2$ in the total system and by $N_A = QlL$ in the subsystem A . Now, let us deform this medium by adding ΔN_A particles without exciting the ground state. This deformation corresponds to the global quench in the medium. Using the regularity of the bulk gauge field $N_A \sim lL\mu^{8/(4-\eta^2)}$, the insertion of particles can be reinterpreted as the change of the chemical potential in a grandcanonical ensemble. For $T_H = 0$, the ground state is not excited so that there is no increment of the entanglement entropy and energy caused by the excited state, $\Delta S_E|_{N_A} = \Delta E|_{N_A} = 0$. Even at zero temperature, however, the global quench can increase the system energy and the entanglement entropy following the thermodynamics-like relation in (77). Since $r_h^4 = N_A^2/(12l^2L^2)$ at zero temperature, the increased energy and entanglement entropy change under the global quench are

$$\begin{aligned} \Delta E|_{T_H} &= \frac{\sqrt[4]{3} \sqrt{N_A}}{\sqrt{2} \kappa^2 \sqrt{l} \sqrt{L}} \Delta N_A, \\ \Delta S_E|_{T_H} &= \frac{\Gamma(\frac{1}{4})^2 (5 \sqrt[4]{12} \pi \sqrt{lLN_A} - 8 lN_A)}{160 \Gamma(\frac{3}{4})^2 \kappa^2 L} \Delta N_A. \end{aligned} \quad (81)$$

In the small l limit, the first term in the entanglement entropy change becomes dominant, so the global quench creates the following energy and entanglement entropy approximately

$$\Delta E|_{T_H} \sim \sqrt{\frac{N_A}{lL}} \Delta N_A,$$

$$\Delta S_E|_{T_H} \sim \sqrt{\frac{l N_A}{L}} \Delta N_A. \quad (82)$$

3.3 Nonconformal medium with a general η

Due to the universal behavior of the entanglement temperature, as mentioned before, it would be interesting to ask whether the size dependence of the entanglement chemical potential is independent of the detail of the theory. In this section, we will check this point and investigate how the global quench in a nonconformal medium changes the entanglement entropy. This result would be useful to figure out the quantum entanglement of the matter states and helpful to understand the real condensed matter system at low temperature.

From (61), for a general η the entanglement entropy in the strip is given by

$$S_E = \frac{4\pi L}{\kappa^2} r_{UV} + s_0 l^{-\frac{4+\eta^2}{4-\eta^2}} + s_1 l^2 + s_2 l^{\frac{4+\eta^2}{4-\eta^2}} + \dots \quad (83)$$

with

$$\begin{aligned} s_0 &= -\frac{\pi^{\frac{8-\eta^2}{4-\eta^2}} (4+\eta^2)^{\frac{4+\eta^2}{4-\eta^2}} \Gamma\left(\frac{3}{4} - \frac{\eta^2}{16}\right)^{\frac{8}{4-\eta^2}} L}{2^{\frac{4+5\eta^2}{4-\eta^2}} \left(\frac{1}{4} - \frac{\eta^2}{16}\right)^{\frac{4+\eta^2}{4-\eta^2}} \Gamma\left(\frac{1}{4} - \frac{\eta^2}{16}\right)^{\frac{8}{4-\eta^2}} \kappa^2}, \\ s_1 &= \frac{(4-\eta^2)^2 \Gamma\left(\frac{1}{4} - \frac{\eta^2}{16}\right)^3 \left[1 + (4+\eta^2) N^2 / \left(16 l^2 L^2 r_h^{\frac{16}{4+\eta^2}}\right)\right] L r_h^{\frac{12-\eta^2}{4+\eta^2}}}{2^{\frac{7}{2} + \frac{\eta^2}{8}} (8-\eta^2) (4+\eta^2) \Gamma\left(1 - \frac{\eta^2}{8}\right) \Gamma\left(\frac{3}{4} - \frac{\eta^2}{16}\right) \kappa^2}, \\ s_2 &= -\frac{\pi^{\frac{4-3\eta^2}{2(4-\eta^2)}} (4-\eta^2)^{\frac{8}{4-\eta^2}} \Gamma\left(\frac{1}{2} - \frac{\eta^2}{8}\right) \Gamma\left(\frac{1}{4} - \frac{\eta^2}{16}\right)^{\frac{4+\eta^2}{4-\eta^2}} N^2}{2^{\frac{64+(4-\eta^2)^2}{8(4-\eta^2)}} (20-\eta^2) (4+\eta^2)^{\frac{4+\eta^2}{4-\eta^2}} \Gamma\left(\frac{3}{4} - \frac{\eta^2}{16}\right)^{\frac{12-\eta^2}{4-\eta^2}} \kappa^2 L}. \end{aligned} \quad (84)$$

Following the same strategy used in the previous section, for a fixed particle number N_A the excitation of the ground states leads to the following entanglement entropy change

$$\begin{aligned} \Delta S_E &= \frac{(4-\eta^2)^2 \Gamma\left(\frac{1}{4} - \frac{\eta^2}{16}\right)^3 l^2 L}{2^{\frac{7}{2} + \frac{\eta^2}{8}} (8-\eta^2) (4+\eta^2) \Gamma\left(1 - \frac{\eta^2}{8}\right) \Gamma\left(\frac{3}{4} - \frac{\eta^2}{16}\right) \kappa^2} \\ &\times \left[r_h^{\frac{12-\eta^2}{4+\eta^2}} \left(1 + \frac{4+\eta^2}{16 l^2 L^2} \frac{N_A^2}{r_h^{\frac{16}{4+\eta^2}}}\right) - r_0^{\frac{12-\eta^2}{4+\eta^2}} \left(1 + \frac{4+\eta^2}{16 l^2 L^2} \frac{N_A^2}{r_0^{\frac{16}{4+\eta^2}}}\right) \right], \end{aligned} \quad (85)$$

where r_0 is the horizon in the extremal limit. This entanglement entropy in the nonconformal medium satisfies the thermodynamics-like relation

$$\Delta E|_{N_A} = T_E \Delta S_E|_{N_A}, \quad (86)$$

with the following entanglement temperature

$$T_E = \frac{2^{\frac{11}{2} + \frac{\eta^2}{8}} (8 - \eta^2) \Gamma\left(1 - \frac{\eta^2}{8}\right) \Gamma\left(\frac{3}{4} - \frac{\eta^2}{16}\right)}{(4 - \eta^2)^2 \Gamma\left(\frac{1}{4} - \frac{\eta^2}{16}\right)^3} \frac{1}{l}. \quad (87)$$

This result shows that the universality of the entanglement temperature also appears in the nonconformal medium.

Now, let us turn to a entanglement chemical potential. The global quench corresponding to the insertion of particles leads to changes of the entanglement entropy and energy. Similar to the previous RNAdS black brane case, these two quantities satisfies the thermodynamics-like relation

$$T_E \Delta S_E|_{T_H} = \Delta E|_{T_H} - \mu_E \Delta N_A, \quad (88)$$

where the increased energy and the entanglement chemical potential are given by

$$\begin{aligned} \Delta E|_{T_H} &= \frac{4 \left[(\eta^2 + 4)^2 N_A^2 - 16 (\eta^2 - 12) l^2 L^2 r_h^{\frac{16}{\eta^2+4}} \right] N_A}{\left[(4 + \eta^2)^2 (12 + \eta^2) N_A^2 + 16 (48 - 16\eta^2 + \eta^4) l^2 L^2 r_h^{\frac{16}{4+\eta^2}} \right] \kappa^2 l L r_h} \Delta N_A \\ \mu_E &= \frac{2^{\frac{32-8\eta^2-\eta^4}{2(4-\eta^2)}} \pi^{\frac{4-2\eta^2}{4-\eta^2}} (8 - \eta^2) (4 - \eta^2)^{\frac{2\eta^2}{4-\eta^2}} \Gamma\left(1 - \frac{\eta^2}{4}\right) N_A}{(20 - \eta^2) (4 + \eta^2)^{\frac{4+\eta^2}{4-\eta^2}} \Gamma\left(\frac{1}{4} - \frac{\eta^2}{16}\right)^{\frac{8-4\eta^2}{4-\eta^2}} \Gamma\left(\frac{3}{4} - \frac{\eta^2}{16}\right)^{\frac{8}{4-\eta^2}} \kappa^2 L} \frac{1}{l^{\frac{2\eta^2}{4-\eta^2}}}. \end{aligned} \quad (89)$$

In the nonconformal medium, the entanglement chemical potential is proportional to the width of the strip with an appropriate power, $\sim l^{-\frac{2\eta^2}{4-\eta^2}}$. This result shows that the size dependence of the entanglement chemical potential crucially depends on the nonconformality unlike the entanglement temperature. Since $\Delta E|_{T_H}$ in (88) is more dominant than $\mu_E \Delta N_A$ in the limit with small l and η , the global quench at zero temperature creates the following entanglement entropy

$$\Delta S_E|_{T_H} \sim \frac{\Delta E|_{T_H}}{T_E} \sim l^{\frac{1}{2} + \frac{\eta^2}{8}} L^{-\frac{1}{2} + \frac{\eta^2}{8}} N_A^{\frac{1}{2} - \frac{\eta^2}{8}} \Delta N_A. \quad (90)$$

4 Conclusion

We have studied the holographic entanglement entropy of a nonconformal medium and its thermodynamics-like relation. In a usual thermal system including the matter, the number of the matter is regarded as a fundamental variable which together with temperature and volume describes a canonical ensemble. In this case, the additional fundamental variable changes the first law of thermodynamics. In the nonconformal medium dual to the charged dilaton black brane, the similar modification happens in the thermodynamics-like relation of the entanglement entropy. To describe such a modification, we need to introduce a corresponding new variable called the entanglement chemical potential. This new

quantity describes the chemical potential caused by the correlation between quantum states. If the entanglement temperature and entanglement chemical potential are defined properly, the entanglement entropy in a thin strip satisfies the modified thermodynamics-like relation. In addition, the entanglement temperature in a nonconformal medium still shows a universal behavior inversely proportional to the size of the subsystem, while the size dependence of the entanglement chemical potential crucially relies on the nonconformality parameter.

In this paper, we further showed that the thermodynamics-like relation can describe the entanglement entropy change of ground and excited states under the insertion of particles. If particles are added uniformly and suddenly, it represents the global deformation called the global quench. Using the regularity of the bulk gauge field, the insertion of particles can be reinterpreted as the change of the chemical potential in a grandcanonical ensemble. If adding particles at zero temperature, the global quench increases the energy and entanglement entropy of ground states as (90). It would be interesting to compare this result with data of the condensed matter system.

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A Thermodynamics of a charged black brane

Assuming that there exist several roots satisfying $f(r) = 0$, the largest root denoted by r_h corresponds to the event horizon. If the intrinsic free parameter η is fixed, the charged black brane geometry can be classified by two black brane hairs, m and Q . From the fact $f(r_h) = 0$, the black hole mass can be rewritten as

$$m = r_h^a + \frac{b}{r_h}, \quad (\text{A.1})$$

so we can describe the charged black brane in terms of r_h and Q instead of m and Q . Furthermore, the Hawking temperature T_H defined by the surface gravity at the horizon is given by

$$T_H = \frac{12 - \eta^2}{4\pi(4 + \eta^2)} r_h^{(4 - \eta^2)/(4 + \eta^2)} \left(1 - \frac{(4 + \eta^2)^2}{16(12 - \eta^2)} \frac{Q^2}{r_h^{16/(4 + \eta^2)}} \right). \quad (\text{A.2})$$

From this relation, r_h can be implicitly reinterpreted as a function of T_H and Q . As a result, we can investigate the thermodynamics of the charged black brane in terms of the Hawking temperature and

charge by using (A.1) and (A.2). The Bekenstein-Hawking entropy S_{BH} is

$$S_{BH} = \frac{2\pi V_2}{\kappa^2} r_h^{8/(4+\eta^2)}, \quad (\text{A.3})$$

where V_2 denotes the regularized volume in (x, y) plane.

Since the black brane provide a well-defined thermodynamic system, the above charged black brane should satisfy the fundamental thermodynamic relation

$$dE = T_H dS_{BH} - P dV_2 + \mu dN, \quad (\text{A.4})$$

where $N = QV_2$ is the total charge. When N and V_2 are fixed, the internal energy E from the Bekenstein-Hawking entropy becomes in terms of r_h

$$E = \int dS_{BH} T_H = \frac{4V_2}{(4+\eta^2)\kappa^2} r_h^{(12-\eta^2)/(4+\eta^2)} \left(1 + \frac{4+\eta^2}{16} \frac{Q^2}{r_h^{16/(4+\eta^2)}} \right). \quad (\text{A.5})$$

In the canonical ensemble, the free energy as a function of T_H , V_2 and N is given by

$$\begin{aligned} F &= E - T_H S_{BH} \\ &= -\frac{(4-\eta^2)V_2}{2(4+\eta^2)\kappa^2} r_h^{(12-\eta^2)/(4+\eta^2)} \left(1 - \frac{(12+\eta^2)(4+\eta^2)}{16(4-\eta^2)} \frac{Q^2}{r_h^{16/(4+\eta^2)}} \right) \end{aligned} \quad (\text{A.6})$$

where r_h should be a function of T_H , N and V_2 . After tedious calculation, one can easily find other thermodynamic quantities like the entropy S , chemical potential μ and pressure P , from the thermodynamic relation of the canonical ensemble

$$S = -\left. \frac{\partial F}{\partial T_H} \right|_{N, V_2} = \frac{2\pi V_2}{\kappa^2} r_h^{8/(4+\eta^2)}, \quad (\text{A.7})$$

$$\mu = \left. \frac{\partial F}{\partial N} \right|_{T_H, V_2} = \frac{N}{2\kappa^2 r_h V_2}, \quad (\text{A.8})$$

$$P = -\left. \frac{\partial F}{\partial V_2} \right|_{T_H, N} = \frac{4-\eta^2}{2(4+\eta^2)\kappa^2} r_h^{(12-\eta^2)/(4+\eta^2)} \left(1 + \frac{4+\eta^2}{16V_2^2} \frac{N^2}{r_h^{16/(4+\eta^2)}} \right). \quad (\text{A.9})$$

These thermodynamic quantities derived from the thermodynamic law are consistent with the previous holographic renormalization results. For example, the Bekenstein-Hawking entropy in (A.3) coincides with the renormalized thermal entropy in (27).

Using the Legendre transformation, it is also possible to describe the charged black brane as a grandcanonical ensemble. In this case, the most important thermodynamic function is the grand potential as a function of T_H , μ and V_2

$$\begin{aligned} \Omega &= F - \mu N \\ &= -\frac{(4-\eta^2)V_2}{2(4+\eta^2)\kappa^2} r_h^{(12-\eta^2)/(4+\eta^2)} \left(1 + \frac{(4+\eta^2)\kappa^4}{4} \frac{\mu^2}{r_h^{2(4-\eta^2)/(4+\eta^2)}} \right), \end{aligned} \quad (\text{A.10})$$

where r_h is a function of T_H and μ only. Then, other thermodynamic quantities are from the thermodynamic law

$$S = - \left. \frac{\partial \Omega}{\partial T_H} \right|_{\mu, V_2} = \frac{2\pi V_2}{\kappa^2} r_h^{8/(4+\eta^2)}, \quad (\text{A.11})$$

$$N = - \left. \frac{\partial \Omega}{\partial \mu} \right|_{T_H, V_2} = 2\kappa^2 V_2 r_h \mu, \quad (\text{A.12})$$

$$P = \frac{(4 - \eta^2)}{2(4 + \eta^2)\kappa^2} r_h^{(12-\eta^2)/(4+\eta^2)} \left(1 + \frac{(4 + \eta^2)\kappa^4}{4} \frac{\mu^2}{r_h^{2(4-\eta^2)/(4+\eta^2)}} \right). \quad (\text{A.13})$$

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